

Effective Field Theory of a Locally Noncommutative Space-Time and Extra Dimensions

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Abstract

We assume that the noncommutativity starts to be visible continuously from a scale Λ_{NC} . According to this assumption, a two-loop effective action is derived for noncommutative ϕ^4 and ϕ^3 theories from a Wilsonian point of view. We show that these effective theories are free of UV/IR mixing phenomena. We also investigate the positivity constraint on coefficients of higher dimension operators present in the effective theory. This constraint makes the low energy theory to be UV completion of a full theory. Finally, we discuss noncommutativity and extra dimensions. In our effective theories formulated on noncommutative extra dimensions, if the compactification scale Λ_c is less than the scale Λ_{NC} , the theory will not suffer from UV/IR mixing.

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1 Introduction

Models of noncommutative space-time have become increasingly popular recently and are believed to be reasonable candidates for Planck scale physics. The main idea is that in the scales where quantum theory and general relativity are no longer independent, the notion of a point in space-time becomes meaningless and a finite minimal length or uncertainty relation will have to be postulated for the coordinate functions in order to prohibit the localization of points with arbitrary high precision. Models with uncertainty relations are usually implemented by considering noncommutative coordinate operators: $[x^\mu, x^\nu] = i\theta^{\mu\nu}$ where $\theta^{\mu\nu}$ is a real and antisymmetric tensor and is of dimension $(mass)^{-2}$. Field theories on noncommutative space-time have received a great deal of attention, not least because they arise naturally in a particular Seiberg-Witten limit [1] of string theory (see [2, 3]) for reviews. Noncommutative field theories are constructed from conventional (commutative) field theory by replacing in the Lagrangian the usual multiplication of fields with the \star -product of fields,

$$(\varphi \star \varphi)(x) = \varphi(x) e^{\frac{i}{2}\theta^{\mu\nu} \overleftarrow{\partial}_\mu \overrightarrow{\partial}_\nu} \varphi(x) \quad (1)$$

The parameter $\theta^{\mu\nu}$ then appears in the vertices of perturbation theory and defines a second mass scale in the theory, called noncommutativity scale Λ_{NC} . In general, for any noncommutative field theory, the loop diagrams can be classified into the so-called “planar” and “non-planar” graphs where the planar part of diagrams show the same UV divergence structure as the corresponding commutative theory, while the non-planar pieces are UV finite. It is natural to ask whether these theories are renormalizable. The UV renormalizability of noncommutative field theories have been investigated using counter term [4, 5, 6, 7] and using Wilson-Polchinski method [8, 9, 10]. On the other hand, perturbation calculations show the noncommutative field theories to be afflicted from an endemic nontrivial mixing of ultraviolet (UV) and infrared (IR) divergences [11]. As a consequence, the Wilsonian approach to field theory seems to break down. Here the high energy modes of the noncommutative theory can not be decoupled because integrating out high energy degrees of freedom produces unexpected low energy divergences induced in the infrared operators of negative dimension [11, 12]. Generally speaking, in every quantum field theory formulated on a noncommutative space-time, we will have problems defining low energy Wilsonian effective action since UV and IR scales do not decouple. This lack of decoupling of different scales in the theory might pose serious problems for phenomenology, since noncommutative effects can show up at low energies interfering with standard model predictions [12, 13, 14]. However, the ordi-

nary soft and collinear divergences are decoupled from noncommutative IR divergences and there is no mixing between them [15]. Recently the renormalizability of noncommutative field theories have been investigated by introducing a harmonic oscillatory term in the free part of Lagrangian. [16, 17, 18, 19]. Here in this work, we propose a new approach to recover the Wilsonian effective action. Our main assumption is that there is an infrared cut-off Λ_{NC} under which noncommutativity can not be probed and space-time becomes commutative continuously. At energies below Λ_{NC} , physics of all degrees of freedom is governed by a commutative theory. The scale Λ_{NC} can be equal to the scale of validity of an ordinary quantum field theory description of nature that is probably around $(1-100)TeV$ [20]. So we can probe a truly noncommutative behavior only in the range $\Lambda_{NC} < k < \Lambda_0$. Here, the ultraviolet cut-off Λ_0 is the scale beyond which the gravitational effects are comparable to those of the rest of the fundamental interactions, i.e. $\Lambda_0 \sim M_P$ (Planck mass). Beyond the scale Λ_0 , the noncommutative theory breaks down and physics is sensitive to the UV completion of the theory. Now the Wilsonian effective action can be obtained for noncommutative ϕ^4 and ϕ^3 by integrating out the noncommutative high energy fields. The feature of these effective theories is that they are free from noncommutative IR divergences or UV/IR mixing. These results can be extended to all orders of g . In the last section, we will discuss the noncommutative extra dimension by employing the ϕ^3 example on $R^{1,3} \times T_\theta^2$. The paper is organized as follows: In section 2, we will first review the renormalizability of noncommutative ϕ^4 theory using Wilson-Polchinski method. Section 3 is devoted to a detailed analysis of deriving Wilsonian effective action for noncommutative ϕ^4 and ϕ^3 theories. We also discuss the positivity constraint [21] on coefficients of higher dimensional terms. In section 4, we discuss the noncommutative extra dimensions and how the UV/IR mixing can be canceled out in the perturbation calculation.

2 Renormalizability of the noncommutative ϕ^4 theory

In this section we first review the renormalization of noncommutative ϕ^4 theory [8, 9, 10] using the Wilsonian flow equation formulated by Polchinski [22, 23]. Then we will have a short review of Grosse and Wulkenhaar method [16, 17, 18, 19]. At first our starting

point is the path integral

$$Z[J] = e^{W[J]} = \int \mathcal{D}\phi \exp \left\{ -\frac{1}{2} \int \frac{d^4 p}{(2\pi)^4} \phi(p) D^{-1} \phi(-p) - S_{\text{int}}[\phi] + \int \frac{d^4 p}{(2\pi)^4} \phi(p) J(-p) \right\} \quad (2)$$

where

$$S_{\text{int}} = \frac{1}{4!} g \int \frac{d^4 p}{(2\pi)^4} \phi \star \phi \star \phi \star \phi, \quad D(p) = (p^2 + m^2)^{-1} \quad (3)$$

Due to star product of fields, the Feynman rules for the interaction vertex changes to

$$\begin{aligned} \Gamma^{(4)}(p_1, p_2, p_3, p_4) &= g^2 h(p_1, p_2, p_3, p_4) \\ &\equiv \frac{g^2}{3} \left[\cos\left(\frac{1}{2} p_1 \theta p_2\right) \cos\left(\frac{1}{2} p_3 \theta p_4\right) + \cos\left(\frac{1}{2} p_1 \theta p_3\right) \cos\left(\frac{1}{2} p_2 \theta p_4\right) \right. \\ &\quad \left. + \cos\left(\frac{1}{2} p_1 \theta p_4\right) \cos\left(\frac{1}{2} p_2 \theta p_3\right) \right] \end{aligned} \quad (4)$$

Working with this vertex leads to some problems. For instance, consider the one loop mass correction

$$\begin{aligned} \Sigma_{\Lambda_0}(p) &= -\frac{g^2}{6} \int \frac{d^4 q}{(2\pi)^4} \Theta(\Lambda_0^2 - q^2) \frac{1}{q^2 + m^2} [2 + \cos(q \cdot \tilde{p})] \\ &= -\frac{g^2}{32\pi^2} \left[\frac{2}{3} \Lambda_0^2 + \frac{4}{3 \tilde{p}^2} (1 - J_0(\Lambda_0 \tilde{p})) \right], \quad (m^2 \ll \min\{\Lambda_0^2, 1/\tilde{p}^2\}) \end{aligned} \quad (5)$$

where $\tilde{p}^\mu \equiv \theta^{\mu\nu} p^\nu$. In equation (5), the second term is contributed by the non-planar graphs containing an extra phase factor and a new kind of IR divergence induced by the effective UV cut-off (UV/IR mixing) arising from this term [11]. To get the RG equation, we first introduce an UV and IR cut-offs Λ_0 and Λ into the theory by making the following substitution

$$D(p) \rightarrow D_{\Lambda, \Lambda_0}(p) \equiv D(p) K_{\Lambda, \Lambda_0}(p) \quad (6)$$

where $K_{\Lambda, \Lambda_0}(p)$ is equal to one in the region $\Lambda^2 < p^2 < \Lambda_0^2$ and vanishes rapidly outside. The substitution above defines Z_{Λ, Λ_0} and W_{Λ, Λ_0} , the generating functionals of Green functions in which only momenta between Λ and Λ_0 have been integrated out. By means of a simple recipe given in [8, 9, 10], we obtain the RG equation for a given Green function. These equations form an infinite system of coupled ordinary differential equations. As an example, the evolution equation for self-energy is given by

$$\Lambda \frac{\partial}{\partial \Lambda} \Sigma_{\Lambda, \Lambda_0}(p) = \frac{1}{2} \int \frac{d^4 q}{(2\pi)^4} \frac{S_{\Lambda, \Lambda_0}(q)}{q^2 + m^2} \Gamma_{\Lambda, \Lambda_0}^4(q, p, -p, -q) \quad (7)$$

Where

$$\begin{aligned} \frac{S_{\Lambda, \Lambda_0}(q)}{q^2 + m^2} &\equiv \Lambda \frac{\partial}{\partial \Lambda} \frac{1}{(q^2 + m^2)K_{\Lambda, \Lambda_0}(q)^{-1} + \Sigma_{\Lambda', \Lambda_0}(q)} \Big|_{\Lambda' = \Lambda}, \\ &= \frac{1}{q^2 + m^2} \frac{1}{\left[1 + \frac{\Sigma_{\Lambda, \Lambda_0}(q)}{q^2 + m^2} K_{\Lambda, \Lambda_0}(q)\right]^2} \Lambda \frac{\partial}{\partial \Lambda} K_{\Lambda, \Lambda_0}(q) \end{aligned} \quad (8)$$

In order to study the UV renormalization, the relevant couplings i.e. those with non-negative mass dimensions, have to be isolated. They are

$$\gamma_2(\Lambda) \equiv \frac{d\Sigma_{\Lambda, \Lambda_0}(p)}{dp^2} \Big|_{p^2=p_0^2}, \quad \gamma_3(\Lambda) \equiv \Sigma_{\Lambda, \Lambda_0}(p) \Big|_{p^2=p_0^2}, \quad \gamma_4(\Lambda) \equiv \frac{\Gamma_{\Lambda, \Lambda_0}^4(\bar{p}_1, \dots, \bar{p}_4)}{h(\bar{p}_1, \dots, \bar{p}_4)} \quad (9)$$

where the momentum \bar{p}_i is chosen such that $\bar{p}_i \cdot \bar{p}_j = p_0^2(\delta_{ij} - \frac{1}{4})$ in which p_0^2 is the renormalization scale. The two- and four-point function can be rewritten as

$$\begin{aligned} \Sigma_{\Lambda, \Lambda_0}(p) &= \gamma_3(\Lambda) + (p^2 - p_0^2)\gamma_2(\Lambda) + \Delta_{\Lambda, \Lambda_0}^2(p) \\ \Gamma_{\Lambda, \Lambda_0}^4(p_1, \dots, p_4) &= \gamma_4(\Lambda) + \Delta_{\Lambda, \Lambda_0}^4(p_1, \dots, p_4) \end{aligned} \quad (10)$$

where $\Delta_{\Lambda, \Lambda_0}^2$ and $\Delta_{\Lambda, \Lambda_0}^4$ are irrelevant operators with all $\Gamma_{\Lambda, \Lambda_0}^{2n}$'s with $n \geq 2$. Integrating RG equations with respect to cut-off parameter from 0 to Λ for the relevant operators and from Λ to Λ_0 for irrelevant operators leads to a set of coupled integral equations with the boundary conditions

$$\gamma_2(0) = 0, \quad \gamma_3(0) = p_0^2, \quad \gamma_4(0) = g^2 \quad (\text{at physical point } \Lambda = 0) \quad (11)$$

and

$$\Delta_{\Lambda_0, \Lambda_0}^2(p) = \Delta_{\Lambda_0, \Lambda_0}^4(p_1, \dots, p_4) = \Gamma_{\Lambda_0, \Lambda_0}^{2n}(p_1, \dots, p_{2n}) = 0 \quad (n > 2) \quad (12)$$

The above boundary conditions indicate the renormalization conditions. For instance, at one loop, the relevant and irrelevant couplings of the two-point functions contribute as in the following:

$$\gamma_3(\Lambda) = -\frac{g^2}{32\pi^2} \left[\frac{2}{3}\Lambda^2 + \frac{4}{3\tilde{p}_0^2}(1 - J_0(\Lambda\tilde{p}_0)) \right], \quad (13)$$

$$\gamma_2(\Lambda) = \frac{g^2}{24\pi^2} \frac{1}{p_0^2\tilde{p}_0^2} \left[1 - J_0(\Lambda\tilde{p}_0) - \frac{\Lambda\tilde{p}_0}{2} J_1(\Lambda\tilde{p}_0) \right], \quad (14)$$

$$\begin{aligned} \Delta_{\Lambda, \Lambda_0}^2 &= \frac{g^2}{24\pi^2} \left\{ \frac{J_0(\Lambda\tilde{p}) - J_0(\Lambda_0\tilde{p})}{\tilde{p}^2} - \frac{J_0(\Lambda\tilde{p}_0) - J_0(\Lambda_0\tilde{p}_0)}{\tilde{p}_0^2} \right. \\ &\quad \left. \frac{p^2 - p_0^2}{p_0^2\tilde{p}_0^2} \left[J_0(\Lambda\tilde{p}_0) - J_0(\Lambda_0\tilde{p}_0) + \frac{\Lambda\tilde{p}_0}{2} J_1(\Lambda\tilde{p}_0) - \frac{\Lambda_0\tilde{p}_0}{2} J_1(\Lambda_0\tilde{p}_0) \right] \right\}. \end{aligned} \quad (15)$$

The proof of renormalizability is based on the scaling behavior of relevant and irrelevant operators. In order to simplify the power counting, we assume the following relations to hold between scales,

$$\tilde{p}\Lambda_0 \gg 1 \quad , \quad \Lambda_{NC} \ll \Lambda \ll \Lambda_0 \quad (16)$$

The UV renormalizability can be proved using induction at any order of perturbation. At one-loop level, the UV cut-off Λ_0 is removed from operators [8, 9, 10]. Now at l -loop, we assume the Λ_0 -independent scaling using power law behavior for operators; for example,

$$\gamma_2^{(l)} \sim \gamma_4^{(l)} \sim |\Delta_{(l)}^4|_\Lambda = O(1) \quad , \quad \gamma_3^{(l)} \sim |\Delta_{(l)}^2|_\Lambda = O(\Lambda^2) \quad (17)$$

Using the renormalization group equation, it can be shown that at $(l+1)$ -loop, the relevant and irrelevant couplings are finite in the $\Lambda_0 \rightarrow \infty$ (UV) limit. So the perturbation renormalizability at any order in perturbation theory is proved. On the other hand, when the IR region ($\tilde{p}\Lambda_0 \ll 1$) comes under scrutiny, the Wilsonian picture breaks down because of the UV/IR phenomena, and the low energy predictions under the scale $\frac{\Lambda_{NC}}{\Lambda_0}$ become highly sensitive to the details of the ultraviolet sector of the theory. In order to write down a Wilsonian effective action which correctly describes the low momentum behavior of the theory, one may add a new degree of freedom χ to the commutative action [11]. Then, the modified Wilsonian action gets the following form

$$S'_{eff}(\Lambda) = S_{eff}(\Lambda) + \int d^4x \left(\frac{1}{2} (\tilde{\partial}\chi)^2 + \frac{1}{2} \Lambda^2 (\tilde{\partial}^2\chi)^2 + i \frac{1}{\sqrt{96\pi}} g\chi\phi \right) \quad (18)$$

Integrating out the new mode in the region $\tilde{p}\Lambda_0 \ll 1$ gives the quadratic infrared singularity $\frac{1}{\tilde{p}^2}$ appearing in the self energy diagrams (5) [8, 9, 10, 11].

There is also a new approach developed by Grosse and Wulkenhaar in [16, 17, 18, 19] where it is shown that noncommutative four dimensional field theories are renormalizable to all orders of perturbation. This method also is a solution to UV/IR mixing problem. This work is performed by adding a new harmonic oscillator term to the free part of the action,

$$S(\phi) = \int d^4x \left(\partial_\mu \phi \star \partial^\mu \phi + \frac{\Omega^2}{2} (\tilde{x}_\mu \phi) \star (\tilde{x}^\mu \phi) + \frac{m^2}{2} \phi \star \phi \right)$$

where $\tilde{x}_\mu = 2(\theta^{-1})_{\mu\nu}x^\nu$ and Ω is a dimensionless constant. This additional new term causes that the theory to be invariant under a sort of duality transformation between positions and momenta at $\Omega = 1$ (away from this special point, it is more precise to say that the model is covariant) [24]. A main difference between this theory and ordinary

noncommutative theories is that a free particle is affected by noncommutativity of space-time i.e. its propagator and also dispersion relation are modified without any interaction. In momentum space, working with this theory and specially deriving the propagators is quite hard and is done only in a matrix representation form. Although if one work in a x-space formalism [25], it seems that to some extent, this cumbersome theory is simplified.

In the following section, we explain a new idea which is a physical realization of mathematical view point of reference [26] and derive an effective action. However, in this paper we do not use a Minkowski space-time and perform our calculation with a Euclidean signature. In this method we introduce a scale where space starts to be noncommutative, and by integrating out the noncommutative fields with momenta beyond this scale, we get an effective theory which is free from UV/IR mixing. The harmonic oscillator term $\frac{\Omega^2}{2}(\tilde{x}_\mu\phi) \star (\tilde{x}^\mu\phi)$ does not change our result because the integration domain begins from a scale where beyond it this term become irrelevant. So we only use the usual noncommutative field theory as we only need to integrate out high energy modes.

3 Noncommutative Wilsonian effective action

3.1 ϕ^4 theory

In ordinary field theory, low-energy dynamics (large distances) does not depend on details of high energy dynamics (short distance). Using Wilson's method, one can decouple the high energy modes and obtain an effective field theory which has the same infrared behaviour as the underlying fundamental theory. In noncommutative field theory, the high energy modes are not decoupled because of UV/IR phenomena and the Wilsonian approach to effective field theory seems to break down. However, making a new assumption we start to rederive a well-behaved Wilsonian effective action for noncommutative ϕ^4 and ϕ^3 theories. At first, we turn to noncommutative ϕ^4 and write the generating functional for it in the d -dimensional Euclidean space. Here we consider the Grosse-Wulkenhaar term,

$$Z = \int \mathcal{D}\phi \exp \left(- \int d^d x \left[\frac{1}{2} \partial_\mu \phi \partial^\mu \phi + m^2 \phi^2 + \frac{\Omega^2}{2} (\tilde{x}_\mu \phi) \star (\tilde{x}^\mu \phi) + \frac{1}{4!} g \phi \star \phi \star \phi \star \phi \right] \right) \quad (19)$$

We now divide the integration variables $\phi(k)$ into two groups, those with $|k| < \Lambda = \Lambda_{NC}$ and those with $\Lambda = \Lambda_{NC} \leq |k| < \Lambda_0$. The main assumption in our calculation is to suppose that the noncommutative behaviour can be detected in the range of

energies beyond Λ_{NC} and that below this scale the theory becomes to be continuously commutative . So we define the fields ϕ with the momentum $|k| < \Lambda_{NC}$ to have an ordinary (commutative) behavior and the fields $\hat{\phi}$ with the momentum $\Lambda_{NC} \leq |k| < \Lambda_0$ to have noncommutative behavior. Then we can replace the old field ϕ by $\phi + \hat{\phi}$.

The Eq.(19) without the harmonic oscillator term is given by,

$$Z = \int \mathcal{D}\phi \int \mathcal{D}\hat{\phi} \exp \left(- \int d^d x \left[\frac{1}{2} (\partial_\mu \phi + \partial_\mu \hat{\phi})^2 + \frac{1}{2} m^2 (\phi + \hat{\phi})^2 + \frac{1}{4!} g (\phi + \hat{\phi})^4 \right] \right) \quad (20)$$

Here by expanding the terms in (20), a new problem will appear. Because of previous assumptions about ϕ and $\hat{\phi}$ we can write

$$\begin{aligned} \phi \star \phi &= \phi \phi \\ \phi \star \hat{\phi} &= \phi \hat{\phi} \\ \hat{\phi} \star \hat{\phi} &= \hat{\phi} \star \hat{\phi} \end{aligned} \quad (21)$$

i.e. the product becomes commutative one whenever one of fields which is being multiplied has a momentum which is less than the noncommutative scale. The ambiguity is appeared when for example we consider a product such as $(\phi + \hat{\phi}) \star (\phi + \hat{\phi}) \star (\phi + \hat{\phi})$. Since

$$(\phi + \hat{\phi}) \star [(\phi + \hat{\phi}) \star (\phi + \hat{\phi})] = [(\phi + \hat{\phi}) \star (\phi + \hat{\phi})] \star (\phi + \hat{\phi}) \quad (22)$$

it does not matter which one we take to expand. If we expand the left side so the terms that are linear in $\hat{\phi}$ are

$$\begin{aligned} \phi \star (\phi \star \hat{\phi}) \\ \phi \star (\hat{\phi} \star \phi) \\ \hat{\phi} \star (\phi \star \phi) \end{aligned} \quad (23)$$

The first two can be evaluated easily according to the rule above and give $\phi^2 \hat{\phi}$. However the last one gives

$$\hat{\phi} \star (\phi \star \phi) = \hat{\phi} \star (\phi^2) \quad (24)$$

Now the modes in ϕ^2 will have momentum which is a sum of those coming from the two ϕ 's. So even if ϕ contain only modes with low momentum, the ϕ^2 will contain modes with momentum larger than the scale Λ_{NC} and so one can not assume ϕ^2 has commutative behavior.

In order to solve this ambiguity it seems that in a more fundamental treatment we must find a symmetry breaking mechanism to deal with the high and low energy modes.

However, in lack of such a theory we may continue with some approximations in our calculations. For example, in the terms such as ϕ^2 we can approximately ignore some modes whose momentum become larger than Λ_{NC} in comparison with high momentum modes of $\hat{\phi}$ and then we have,

$$Z = \int \mathcal{D}\phi e^{-\int \mathcal{L}(\phi)} \int \mathcal{D}\hat{\phi} \exp \left(- \int d^d x \left[\frac{1}{2} (\partial_\mu \hat{\phi})^2 + \frac{1}{2} m^2 \hat{\phi}^2 + \frac{\Omega^2}{2} (\tilde{x}_\mu \hat{\phi}) \star (\tilde{x}^\mu \hat{\phi}) \right. \right. \\ \left. \left. + g \left(\frac{1}{6} \phi^3 \hat{\phi} + \frac{1}{4} \phi^2 \hat{\phi} \star \hat{\phi} + \frac{1}{6} \phi \hat{\phi} \star \hat{\phi} \star \hat{\phi} + \frac{1}{4!} \hat{\phi} \star \hat{\phi} \star \hat{\phi} \star \hat{\phi} \right) \right] \right) \quad (25)$$

where $\mathcal{L}(\phi)$ is the commutative Lagrangian. Now we can ignore the $\frac{\Omega^2}{2} (\tilde{x}_\mu \hat{\phi}) \star (\tilde{x}^\mu \hat{\phi})$ term since this term is highly irrelevant in short distances or equivalently in energies above Λ_{NC} .

Integrating out all $\hat{\phi}$ s with the momentum $\Lambda_{NC} \leq |k| < \Lambda_0$, transforms (25) into the following form

$$Z = \int \mathcal{D}\phi e^{-S_{eff}(\phi; \Lambda_{NC})} \quad (26)$$

where $S_{eff}(\phi; \Lambda_{NC})$ is the deformed Wilsonian effective action. The corresponding Lagrangian density will finally be rewritten as follows:

$$\mathcal{L}_{eff}(\phi; \Lambda_{NC}) = \frac{1}{2} Z(\Lambda_{NC}) \partial_\mu \phi \partial^\mu \phi + \frac{1}{2} m^2(\Lambda_{NC}) \phi^2 + \frac{1}{4!} g(\Lambda_{NC}) \phi^4 \\ + \sum_i C_{n,d,i}(\Lambda_{NC}) \mathcal{O}_{n,d,i} \quad (27)$$

where the terms $\mathcal{O}_{n,d,i}$ are all irrelevant local operators with the dimension $D = 2n + d \geq 6$ and consist of all the terms that can be constructed out of an even number $2n$ of ϕ fields with d number of derivatives acting on them. The index i keeps track of operators with the same values of n and d that are not equivalent after integration by parts. We will show explicitly the dependence of coefficients Z , m^2 , g and $C_{n,d,i}$'s on the noncommutative parameter θ . The crucial feature of (27) is that the UV/IR mixing problem is solved naturally in it. Also the coefficients $C_{n,d,i}$ satisfy the positivity constraint discussed in [21] in order for the low energy effective theory to be UV complete into a full theory. Now before turning to calculation parameters, note that we assume $m^2 \ll \Lambda_{NC}^2$ and treat the mass term $\frac{1}{2} m^2 \hat{\phi}$ as a perturbation. To perform the perturbation calculation, we expand the exponential and use the Wick's theorem. In a Feynman diagrams approach, the fields ϕ just contribute as an external line and the fields $\hat{\phi}$ contribute via internal lines and its propagator is defined as the commutative ones,

$$\langle \hat{\phi}(p) \hat{\phi}(k) \rangle = \frac{1}{k^2} (2\pi)^d \delta^d(k + p) \Theta(k) \quad (28)$$

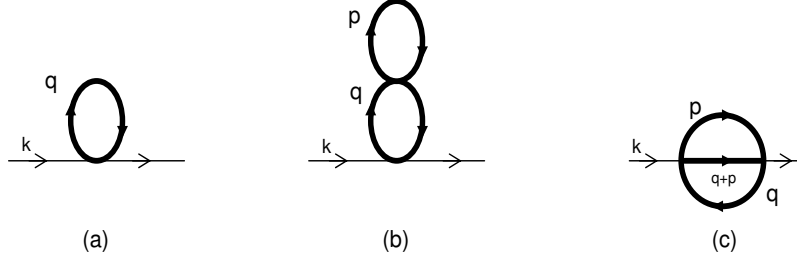


Figure 1: Two external line diagrams

where

$$\Theta(k) = \begin{cases} 1 & \text{if } \Lambda_{NC} \leq |k| < \Lambda_0 \\ 0 & \text{otherwise} \end{cases} \quad (29)$$

The diagrams that must be evaluated for the case of the two external lines are drawn up to order g^2 in Fig. 1. The thin line represents the external fields ϕ and the dark line shows the internal fields $\hat{\phi}$.

First, we calculate Fig. 1a resulting from expansion of $\phi^2 \hat{\phi}^2$ term in the exponent (25). We have

$$-\frac{1}{2} \int d^d x \mu_1 \phi^2 \quad (30)$$

Where

$$\mu_1 = \frac{g}{2} \int \frac{d^d q}{(2\pi)^d} \frac{1}{q^2} = \frac{\pi^{\frac{d}{2}} g}{(2\pi)^d \Gamma(\frac{d}{2})} \frac{1}{d-2} \left(\Lambda_0^{d-2} - \Lambda_{NC}^{d-2} \right) \quad (31)$$

The second diagram (Fig. 1b) results from the term $(\phi^2 \hat{\phi} \star \hat{\phi})(\hat{\phi} \star \hat{\phi} \star \hat{\phi} \star \hat{\phi})$. Then for this diagram we have

$$-\frac{1}{2} \int d^d x \mu_2 \phi^2 \quad (32)$$

Where

$$\begin{aligned} \mu_2 &= \frac{1}{2} \frac{g^2}{4!} \int \frac{d^d p}{(2\pi)^d} \frac{1}{p^2} \int \frac{d^d q}{(2\pi)^d} \frac{1}{(q^2)^2} \\ &= \frac{1}{2} \frac{1}{4!} \left(\frac{2\pi^{\frac{d}{2}} g}{(2\pi)^d \Gamma(\frac{d}{2})} \right)^2 \frac{1}{d-2} \frac{1}{d-4} \left(\Lambda_0^{d-2} - \Lambda_{NC}^{d-2} \right) \left(\Lambda_0^{d-4} - \Lambda_{NC}^{d-4} \right) \end{aligned} \quad (33)$$

In the same way, for the remaining diagram (Fig. 1c) we will have

$$Fig. 1c = -\frac{1}{2} \int d^d x \mu_3 \phi^2 \quad (34)$$

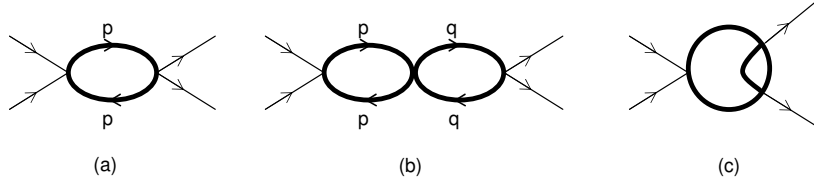


Figure 2: Four external line diagrams

where using the table of integrals [27], calculations will yield:

$$\begin{aligned}\mu_3 &= \left(\frac{g}{6}\right)^2 \int \frac{d^d p}{(2\pi)^d} \frac{1}{p^2} \int \frac{d^d q}{(2\pi)^d} \frac{1 + \cos p \theta q}{q^2 (q + p)^2} \\ &= \left(\frac{g}{6}\right)^2 [\mathcal{F}(1, 1) + \mathcal{G}(1, 1; \theta)]\end{aligned}\quad (35)$$

The functions $\mathcal{F}(1, 1)$ and $\mathcal{G}(1, 1; \theta)$ can be found in the appendix A. In the above equation, the factor $\cos(p\theta k)$ is arising from Feynman rules derived from noncommutative $\hat{\phi}^3$ vertex. Note that at this stage, in Fig. 1c, we have considered the limit in which the external momenta carried out by the factors ϕ are very small compared to Λ_{NC} and they can, therefore, be ignored. The coefficients μ_i give corrections to the m^2 term in \mathcal{L} along the following lines:

$$\begin{aligned}m^2(\Lambda_{NC}) &= m^2(\Lambda_0) + \mu_1 + \mu_2 + \mu_3 \\ &= m^2(\Lambda_0) \\ &\quad + \frac{2\pi^{\frac{d}{2}} g(\Lambda_0)}{(2\pi)^d \Gamma(\frac{d}{2})} \frac{1}{d-2} \left(\Lambda_0^{d-2} - \Lambda_{NC}^{d-2} \right) \\ &\quad + \frac{1}{2} \frac{1}{4!} \left(\frac{2\pi^{\frac{d}{2}} g(\Lambda_0)}{(2\pi)^d \Gamma(\frac{d}{2})} \right)^2 \frac{1}{d-2} \frac{1}{d-4} \left(\Lambda_0^{d-2} - \Lambda_{NC}^{d-2} \right) \left(\Lambda_0^{d-4} - \Lambda_{NC}^{d-4} \right) \\ &\quad + \frac{g^2(\Lambda_0)}{36} [\mathcal{F}(1, 1) + \mathcal{G}(1, 1; \theta)]\end{aligned}\quad (36)$$

Let us now evaluate the diagrams with four external lines, up to order g^3 as drawn in Fig. 2. The first loop diagram is contributed as follows

$$Fig. 2a = -\frac{1}{4!} \int d^d x \zeta_1 \phi^4 \quad (37)$$

where

$$\begin{aligned}
\zeta_1 &= 4! \left(\frac{g}{4}\right)^2 \int \frac{d^d p}{(2\pi)^d} \frac{1}{(p^2)^2} = \frac{3\pi^{\frac{d}{2}} g^2}{(2\pi)^d \Gamma(\frac{d}{2})} \frac{1}{d-4} (\Lambda_0^{d-4} - \Lambda_{NC}^{d-4}) \\
&= \frac{3g^2}{16\pi^2} \log \frac{\Lambda_0}{\Lambda_{NC}} \quad (d \rightarrow 4)
\end{aligned} \tag{38}$$

Similarly, for the remaining two loop diagrams, we have

$$Fig. 2b = -\frac{1}{4!} \int d^d x \zeta_2 \phi^4 \tag{39}$$

where

$$\begin{aligned}
\zeta_2 &= \frac{g^3}{16} \int \frac{d^d p}{(2\pi)^d} \frac{1}{(p^2)^2} \int \frac{d^d p}{(2\pi)^d} \frac{1}{(p^2)^2} \\
&= \frac{g^3}{4} \left[\frac{\pi^{\frac{d}{2}} g^2}{(2\pi)^d \Gamma(\frac{d}{2})} \frac{1}{d-4} (\Lambda_0^{d-4} - \Lambda_{NC}^{d-4}) \right]^2
\end{aligned} \tag{40}$$

and

$$Fig. 2c = -\frac{1}{4!} \int d^d x \zeta_3 \phi^4 \tag{41}$$

with

$$\begin{aligned}
\zeta_3 &= \frac{g^3}{12} \int \frac{d^d p}{(2\pi)^d} \frac{1}{(p^2)^2} \int \frac{d^d q}{(2\pi)^d} \frac{1 + \cos p \cdot \theta q}{q^2 (q+p)^2} \\
&= \frac{g^3}{12} [\mathcal{F}(2, 1) + \mathcal{G}(2, 1; \theta)]
\end{aligned} \tag{42}$$

The coefficients ζ_1 , ζ_2 and ζ_3 give a nonzero correction to g ,

$$\begin{aligned}
g(\Lambda_{NC}) &= g(\Lambda_0) + \zeta_1 + \zeta_2 + \zeta_3 \\
&= g(\Lambda_0) + \frac{3\pi^{\frac{d}{2}} g^2(\Lambda_0)}{(2\pi)^d \Gamma(\frac{d}{2})} \frac{1}{d-4} (\Lambda_0^{d-4} - \Lambda_{NC}^{d-4}) \\
&\quad + \frac{g^3}{4} \left[\frac{\pi^{\frac{d}{2}} g^2}{(2\pi)^d \Gamma(\frac{d}{2})} \frac{1}{d-4} (\Lambda_0^{d-4} - \Lambda_{NC}^{d-4}) \right]^2 \\
&\quad + \frac{g^3}{12} [\mathcal{F}(2, 1) + \mathcal{G}(2, 1; \theta)]
\end{aligned} \tag{43}$$

Dependence of the factors m^2 and g to the noncommutative parameter θ comes from $\mathcal{G}(n, m; \theta)$. This function has a complicate form but for $d = 4$, after performing the summation on r it is possible to see explicitly the soft behaviour of this function in

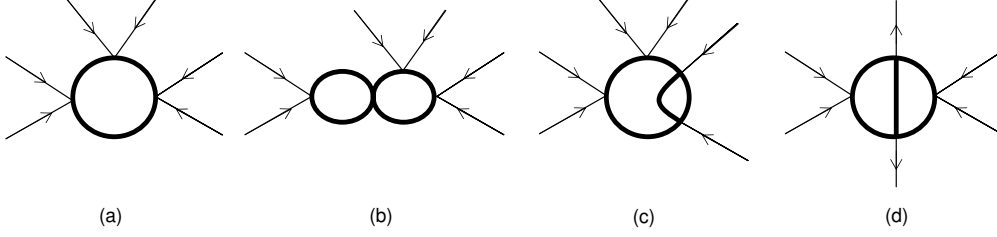


Figure 3: Six external line diagrams

the limit $\theta \rightarrow 0$. So the loop calculation is free of noncommutative IR divergences or UV/IR mixing which is expected as a result of maintaining the assumption that noncommutative effects appear to be visible from the scale Λ_{NC} . Following this formalism, one can extend these results to arbitrary orders of perturbation theory. We now turn to terms with higher orders of ϕ . For instance, the dominant contribution to $C_{6,d,i}(\Lambda_{NC})$ is given by one and two loop diagrams with six external lines, (See Fig. 3). Calculating these diagrams, we find the coefficients $C_{6,d,i}(\Lambda)$ with a negative mass dimension. In this part, the diagrams in Fig. 3 are calculated,

$$Fig. 3a = -\frac{1}{6!} \int d^d x \lambda_1 \phi^6 \quad (44)$$

$$Fig. 3b = -\frac{1}{6!} \int d^d x \lambda_2 \phi^6 \quad (45)$$

$$Fig. 3c = -\frac{1}{6!} \int d^d x \lambda_3 \phi^6 \quad (46)$$

$$Fig. 3d = -\frac{1}{6!} \int d^d x \lambda_4 \phi^6 \quad (47)$$

Where $\lambda_1, \dots, \lambda_4$ are defined as follows

$$\begin{aligned} \lambda_1 &\equiv C_{6,0,1} = 6! \left(\frac{g}{4}\right)^3 \int \frac{d^d q}{(2\pi)^d} \left(\frac{1}{q^2}\right)^3 \\ &= 6! \left(\frac{g}{4}\right)^3 \frac{2\pi^{\frac{d}{2}}}{(2\pi)^d \Gamma(\frac{d}{2})} \frac{1}{d-6} (\Lambda_0^{d-6} - \Lambda_{NC}^{d-6}) \end{aligned} \quad (48)$$

$$\begin{aligned} \lambda_2 &\equiv C_{6,0,2} = 2.6! \frac{g^4}{4!4^3} \int \frac{d^d p}{(2\pi)^d} \frac{1}{(p^2)^2} \int \frac{d^d q}{(2\pi)^d} \frac{1}{(q^2)^3} \\ &= 2.6! \frac{g^4}{4!4^3} \left(\frac{2\pi^{\frac{d}{2}}}{(2\pi)^d \Gamma(\frac{d}{2})}\right)^2 \frac{1}{d-4} \frac{1}{d-6} (\Lambda_0^{d-4} - \Lambda_{NC}^{d-4})(\Lambda_0^{d-6} - \Lambda_{NC}^{d-6}) \end{aligned} \quad (49)$$

$$\begin{aligned}
\lambda_3 \equiv C_{6,0,3} &= \frac{5}{2} g^4 \int \frac{d^d p}{(2\pi)^d} \frac{1}{(p^2)^3} \int \frac{d^d q}{(2\pi)^d} \frac{1 + \cos p \cdot \theta q}{q^2 (q + p)^2} \\
&= \frac{5}{2} g^4 [\mathcal{F}(3, 1) + \mathcal{G}(3, 1; \theta)]
\end{aligned} \tag{50}$$

$$\begin{aligned}
\lambda_4 \equiv C_{6,0,4} &= \frac{5}{2} g^4 \int \frac{d^d p}{(2\pi)^d} \frac{1}{(p^2)^2} \int \frac{d^d q}{(2\pi)^d} \frac{1 + \cos p \cdot \theta q}{(q^2)^2 (q + p)^2} \\
&= \frac{5}{2} g^4 [\mathcal{F}(2, 2) + \mathcal{G}(2, 2; \theta)]
\end{aligned} \tag{51}$$

In addition to terms with the coefficient $C_{n,0,i}(\Lambda_{NC})$, i.e, the terms with nonderivative higher power of ϕ , the higher derivative terms could contribute to the effective Lagrangian. These terms arise in a more exact treatment when we stop neglecting the external momenta of the diagrams. For instance, expanding ϕ in (25), we could obtain the following terms:

$$\int d^d x \, \eta_1 \phi (\partial \phi)^3 \, , \, \int d^d x \, \eta_2 \phi^2 (\partial \phi)^2 \, , \, \int d^d x \, \eta_3 (\partial \phi)^4 \, , \, \dots \tag{52}$$

In general, the procedure of integrating out the fields $\hat{\phi}$ generate all possible interactions of the fields and their derivatives. The coefficients $\eta_1, \eta_2, \eta_3, \dots$ are proportional to

$$\eta_1, \eta_2, \eta_3, \dots \propto \zeta_1 + \zeta_2 + \zeta_3 \tag{53}$$

Usually only nonderivative higher dimensional operators are considered, while higher derivatives are less studied or less popular due to the many difficult issues involved: either unitarity or causality violation, non-locality and presence of ghost fields with superluminal velocity [21, 28, 29, 30]. As already discussed in [21], one can put a positivity constraint on higher order irrelevant operators in order for the low energy effective theory to be UV complete into a full theory correctly. At one loop level, we see this positivity satisfied for the non-derivative terms and we have,

$$C_{n,d,i} > 0 \tag{54}$$

We must note that the positivity constraint is different from other familiar positivity constraints that follow from vacuum stability (for example, the positivity of the kinetic term and also $m^2 \phi^2$ and $g \phi^4$ couplings). The effective models with irrelevant terms and a negative sign can not arise as a low energy limit of any familiar UV-complete theories. The positivity constraint at least at one loop level is satisfied for the irrelevant derivative terms as is explicit from (38) and the fact (48).

3.2 ϕ^3 theory

We repeat the computations performed above for noncommutative ϕ^3 theory whose generating functional in d-dimensional Euclidean space is

$$Z = \int \mathcal{D}\phi \exp \left(- \int d^d x \left[\frac{1}{2} \partial_\mu \phi \partial^\mu \phi + \frac{1}{2} m^2 \phi^2 + \frac{1}{3!} g \phi \star \phi \star \phi \right] \right) \quad (55)$$

It can be shown that in six dimensions the 1PI two point function receives contribution from a one loop nonplanar diagrams for $\tilde{p}\Lambda_0 \ll 1$ and gets the one loop effective action as [11]

$$S_{eff} = \int d^6 p \frac{1}{2} \left(p^2 + M^2 - \frac{g^2}{2^8 \pi^2 \tilde{p}^2} + \frac{g^2}{3 \cdot 2^9 \pi^3} (p^2 + 6M^2) \ln\left(\frac{1}{M^2 \tilde{p}^2}\right) + \dots \right) \phi(p) \phi(-p) \quad (56)$$

Where M^2 is the mass renormalized by the planar diagram. The quadratic IR divergence can be interpreted again in terms of new degrees of freedom χ as in [11]. In spite of this, we try to rederive Wilsonian effective action from our new formalism. So, as in ϕ^4 theory, we divide the variables into commutative fields ϕ and noncommutative fields $\hat{\phi}$. Then using the similar discussions of the previous section, (55) gets the form,

$$Z = \int \mathcal{D}\phi e^{-\int \mathcal{L}(\phi)} \int \mathcal{D}\hat{\phi} \exp \left(- \int d^d x \left[\frac{1}{2} (\partial_\mu \hat{\phi})^2 + \frac{1}{2} m^2 \hat{\phi} \star \hat{\phi} + g \left(\frac{1}{2} \phi \hat{\phi} \star \hat{\phi} + \frac{1}{2} \phi^2 \hat{\phi} + \frac{1}{3!} \hat{\phi} \star \hat{\phi} \star \hat{\phi} \right) \right] \right) \quad (57)$$

Integrating out the high momentum noncommutative degrees of freedom $\hat{\phi}$, we obtain the Wilsonian effective action $S_{eff}(\phi; \Lambda_{NC})$ with the corresponding Lagrangian density,

$$\begin{aligned} \mathcal{L}_{eff}(\phi; \Lambda_{NC}) = & \frac{1}{2} Z(\Lambda_{NC}) \partial_\mu \phi \partial^\mu \phi + \frac{1}{2} m^2(\Lambda_{NC}) \phi^2 + \frac{1}{3!} g(\Lambda_{NC}) \phi^3 \\ & + \sum_i C_{n,d,i}(\Lambda_{NC}) \mathcal{O}_{n,d,i} \end{aligned} \quad (58)$$

We can expand the exponential and use Wick's theorem, to define the propagator and find Feynman rules. The fields $\hat{\phi}$ contribute to internal lines and its propagator will be the same as (28). The diagrams with two, three, and four external lines are drawn in Figs. 4, 5 and 6 respectively. Here we present the expressions associated with the above diagrams in the following form:

$$Fig. 4 = -\frac{1}{2} \int d^d x (\mu_1 + \mu_2 + \mu_3) \phi^2 \quad (59)$$

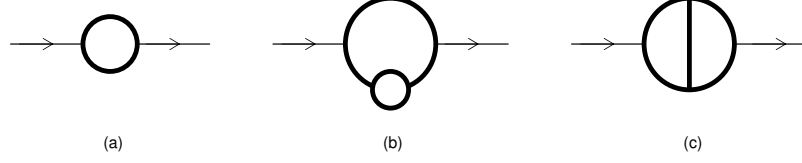


Figure 4: Two external line diagrams

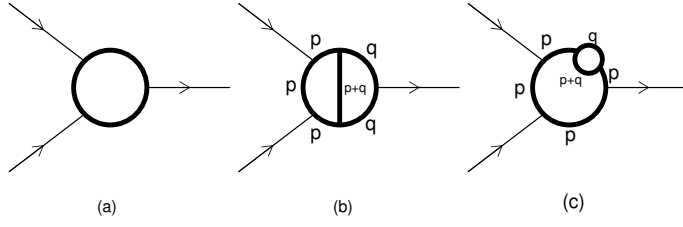


Figure 5: Three external line diagrams

$$Fig. 5 = -\frac{1}{3!} \int d^d x (\zeta_1 + \zeta_2 + \zeta_3) \phi^3 \quad (60)$$

$$Fig. 6 = -\frac{1}{4!} \int d^d x (\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4) \phi^4. \quad (61)$$

In Eq. (59), the terms μ_i are calculated straightforwardly and the results are

$$\mu_1 = \frac{g^2}{2} \int \frac{d^d q}{(2\pi)^d} \frac{1}{(q^2)^2} = g^2 \frac{\pi^{\frac{d}{2}}}{(2\pi)^d \Gamma(\frac{d}{2})} \frac{1}{d-4} (\Lambda_0^{d-4} - \Lambda_{NC}^{d-4}) \quad (62)$$

$$\begin{aligned} \mu_2 &= \frac{g^4}{2 \cdot (3!)^2} \int \frac{d^d p}{(2\pi)^d} \frac{1}{(p^2)^3} \int \frac{d^d q}{(2\pi)^d} \frac{1 + \cos p\theta q}{q^2 (q+p)^2} \\ &= \frac{g^4}{2 \cdot (3!)^2} [\mathcal{F}(3, 1) + \mathcal{G}(3, 1; \theta)] \end{aligned} \quad (63)$$

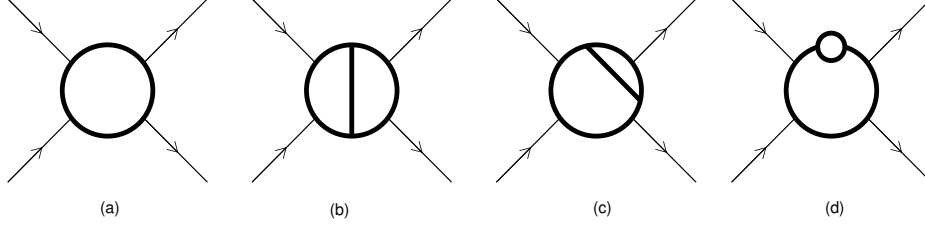


Figure 6: Four external line diagrams

$$\begin{aligned}
\mu_3 &= \frac{g^4}{2 \cdot (3!)^2} \int \frac{d^d p}{(2\pi)^d} \frac{1}{(p^2)^2} \int \frac{d^d q}{(2\pi)^d} \frac{1 + \cos p\theta q}{(q^2)^2 (q+p)^2} \\
&= \frac{g^4}{2 \cdot (3!)^2} [\mathcal{F}(2, 2) + \mathcal{G}(2, 2; \theta)]
\end{aligned} \tag{64}$$

These coefficients modify the m^2 term of \mathcal{L} in (57) as,

$$m^2(\Lambda_{NC}) = m^2(\Lambda_0) + \mu_1 + \mu_2 + \mu_3 \tag{65}$$

Similarly, for the terms ζ_1, ζ_2 and ζ_3 we have,

$$\zeta_1 = 3! \frac{g^3}{8} \int \frac{d^d q}{(2\pi)^d} \frac{1}{(q^2)^3} = \frac{3}{4} g^3 \frac{2\pi^{\frac{d}{2}}}{(2\pi)^d \Gamma(\frac{d}{2})} \frac{1}{d-6} (\Lambda_0^{d-6} - \Lambda_{NC}^{d-6}) \tag{66}$$

$$\begin{aligned}
\zeta_2 &= 3 \frac{g^5}{16 \cdot 3!} \int \frac{d^d p}{(2\pi)^d} \frac{1}{(p^2)^3} \int \frac{d^d q}{(2\pi)^d} \frac{1 + \cos p\theta q}{(q^2)^2 (q+p)^2} \\
&= 3 \frac{g^5}{16 \cdot 3!} [\mathcal{F}(3, 2) + \mathcal{G}(3, 2; \theta)]
\end{aligned} \tag{67}$$

$$\begin{aligned}
\zeta_3 &= 3 \frac{g^5}{16 \cdot 3!} \int \frac{d^d p}{(2\pi)^d} \frac{1}{(p^2)^4} \int \frac{d^d q}{(2\pi)^d} \frac{1 + \cos p\theta q}{q^2 (q+p)^2} \\
&= 3 \frac{g^5}{16 \cdot 3!} [\mathcal{F}(4, 1) + \mathcal{G}(4, 1; \theta)]
\end{aligned} \tag{68}$$

The terms ζ_1, ζ_2 and ζ_3 give a correction to the coupling g as,

$$g(\Lambda_{NC}) = g(\Lambda_0) + \zeta_1 + \zeta_2 + \zeta_3 \tag{69}$$

Note how the effects of noncommutativity appear in the corrections of m^2 and g . Here as in the ϕ^4 theory, in m^2 and g , the quantum correction is free of UV/IR mixing at the limit $\theta \rightarrow 0$. The coefficients of higher order operators (for example, ϕ^4) can be derived by taking into account the diagrams drawn in Fig. 6. For the coefficients λ_i , we have

$$\begin{aligned}\lambda_1 \equiv C_{4,0,1} &= \frac{g^4}{3!} \int \frac{d^d q}{(2\pi)^d} \frac{1}{(q^2)^4} \\ &= \frac{g^4}{3!} \frac{2\pi^{\frac{d}{2}}}{(2\pi)^d \Gamma(\frac{d}{2})} \frac{1}{d-8} (\Lambda_0^{d-8} - \Lambda_{NC}^{d-8})\end{aligned}\quad (70)$$

$$\begin{aligned}\lambda_2 \equiv C_{4,0,2} &= \frac{g^6}{2 \cdot 3!} \int \frac{d^d p}{(2\pi)^d} \frac{1}{(p^2)^3} \int \frac{d^d q}{(2\pi)^d} \frac{1 + \cos p\theta q}{(q^2)^3 (q+p)^2} \\ &= \frac{g^6}{2 \cdot 3!} [\mathcal{F}(3,3) + \mathcal{G}(3,3;\theta)]\end{aligned}\quad (71)$$

$$\begin{aligned}\lambda_3 \equiv C_{4,0,3} &= \frac{g^6}{3!} \int \frac{d^d p}{(2\pi)^d} \frac{1}{(p^2)^4} \int \frac{d^d q}{(2\pi)^d} \frac{1 + \cos p\theta q}{(q^2)^2 (q+p)^2} \\ &= \frac{g^6}{3!} [\mathcal{F}(4,2) + \mathcal{G}(4,2;\theta)]\end{aligned}\quad (72)$$

$$\begin{aligned}\lambda_4 \equiv C_{4,0,4} &= \frac{g^6}{3!} \int \frac{d^d p}{(2\pi)^d} \frac{1}{(p^2)^5} \int \frac{d^d q}{(2\pi)^d} \frac{1 + \cos p\theta q}{q^2 (q+p)^2} \\ &= \frac{g^6}{3!} [\mathcal{F}(5,1) + \mathcal{G}(5,1;\theta)]\end{aligned}\quad (73)$$

In addition to the terms (59-61), we may have some higher dimensional derivative terms. For instance, by expanding the fields ϕ in (65) the expressions,

$$\int d^d x \eta_1 \phi (\partial \phi)^2, \quad \int d^d x \eta_2 \phi^2 \partial \phi, \quad \int d^d x \eta_3 (\partial \phi)^3, \quad \int d^d x \eta_4 \phi^2 \partial^2 \phi, \dots \quad (74)$$

may contribute to (57) with the coefficients η_i proportional to $\zeta_1 + \zeta_2 + \zeta_2$. The positivity constraint [21] on the coefficients of the derivative interaction terms and also higher order operators are concluded from (64) and (70) up to one loop level.

At the end of this section we briefly explain our method and compare it with the theory of Grosse and Wulkenhaar [16, 17, 18, 19]. The approach of the present work is based on the fact that noncommutativity is an effective picture of space-time in the Planck scales where quantum gravity fluctuations change the classical notion of

space and time. So we expect that noncommutativity is only relevant at very high energies while in our ordinary scales, space-time is considered to be commutative. In [16, 17, 18, 19] the term $\frac{\Omega^2}{2}(\tilde{x}_\mu\phi) \star (\tilde{x}^\mu\phi)$ has been considered in order to get rid of UV/IR mixing problem. This term has important effects only at large distances and is irrelevant at short distances. So the low energy theory is affected by noncommutativity but interactions can cure this problem and finally leads the theory be free from UV/IR mixing problem. We have shown that the same result can be obtained by assuming a commutative space at low energies and by integrating out the high energy noncommutative modes beyond the scale Λ_{NC} . In the high energy region existence of the harmonic oscillator term does not change the results of two previous subsection.

4 Noncommutative extra dimension

We first discuss about extra dimension proposed in reference [31]. In a noncommutative theory in addition to noncommutative quadratic IR singularities, there exist noncommutative logarithmic singularities. The authors of [31] have suggested some new light degrees of freedom to interpret these logarithmic IR singularities. For example, in ϕ^3 theory the logarithmically singular behavior of one loop effective action present in expression (56) may be reproduced as a Wilsonian effective action arising from the exchange of new scalar particles χ_1 and χ_2 which couple to fields ϕ and have the propagators,

$$\begin{aligned}\langle\chi_1(p)\chi_2(-p)\rangle &= -\frac{1}{3 \cdot 2^9 \pi^3} \ln\left(\frac{\tilde{p}^2 + \frac{1}{\Lambda^2}}{\tilde{p}^2}\right) \\ \langle\chi_1(p)\chi_1(-p)\rangle &= \langle\chi_2(p)\chi_2(-p)\rangle = 0\end{aligned}\tag{75}$$

The χ_i fields may be replaced with the continuums of states ψ_m which have the ordinary propagators,

$$\langle\psi_m(p)\psi_m(-p)\rangle \propto \frac{1}{\frac{\tilde{p}^2}{(\alpha')^2} + m^2}\tag{76}$$

where the constant α' is defined as $g^{\mu\nu} = -\frac{(\theta^2)^{\mu\nu}}{(\alpha')^2}$. An interpretation for these continuums of degrees of freedom ψ_m is that they are the transverse momentum modes of a particle ψ which propagates freely in extra dimensions. It may be imagined that a d-dimensional space in which the ϕ particles propagate is a flat d-dimensional brane residing in a d+n dimensional space. The ψ particles propagate freely in d+n dimensional space but couple to the ϕ particles on the brane (located at $x_\perp = 0$). Actually the theory with free ψ particles in extra dimensions and coupling directly to ϕ s on

the brane is exactly equivalent to a theory with particles χ that live on the brane and have logarithmic propagators [31]. These extra dimensions may be real or simply a mathematical convenience. In spite of this, it is not necessary in our approach to introduce extra dimensions since the Wilsonian effective action that we rederived is free from noncommutative IR singularities. Of course, even without continuum states ψ_m , the world may have one or more compactified extra dimensions. It is possible that such compact dimensions are noncommutating [32, 33]. In the rest of this section, we will discuss these noncommutative extra dimensions in ϕ^3 theory on $R^{1,3} \times T_\theta^2$ ([32]) where T_θ^2 is a noncommutative two-torus whose coordinates x^4 and x^5 satisfy

$$[x^4, x^5] = i\theta. \quad (77)$$

Since coordinates x^4 and x^5 have been compactified on a rectangular torus with boundary conditions $0 \leq x^4, x^5 \leq 2\pi R$, the momentum along the compact dimensions are quantized as $\mathbf{p} = \frac{\mathbf{n}}{R}$ where $\mathbf{n} = (n_4, n_5)$ are integers. To perform the perturbative calculation, it is better to separate field ϕ as the commutative fields ϕ_0 containing $\mathbf{n} = 0$ and the noncommutative fields $\bar{\phi}$ containing all modes with nonvanishing \mathbf{n} . We make this separation as follows,

$$\phi_0 = \frac{1}{(2\pi R)^2} \int dx^4 dx^5 \phi, \quad (78)$$

and

$$\bar{\phi} = \phi - \phi_0. \quad (79)$$

In terms of these definitions, the partition function (57) in six dimensions gets the form,

$$Z = \int \mathcal{D}\phi \exp \left(- \int d^6x \left[\frac{1}{2}(\partial\bar{\phi})^2 - \frac{1}{2}m^2\bar{\phi}^2 + \frac{1}{2}(\partial\phi_0)^2 - \frac{1}{2}m^2\phi_0^2 - \frac{g}{3!}\bar{\phi} \star \bar{\phi} \star \bar{\phi} - \frac{g}{2}\phi_0\bar{\phi} \star \bar{\phi} - \frac{g}{3!}\phi_0^3 \right] \right). \quad (80)$$

Note that the relation between compactification scale $\Lambda_c = \frac{1}{R}$ and Λ_{NC} is unknown. Naturally, there are two possibilities about their relation: $\Lambda_c < \Lambda_{NC}$ or $\Lambda_c \geq \Lambda_{NC}$. If $\Lambda_c < \Lambda_{NC}$ (for example $\frac{1}{R} < \text{few TeV}$), the extra dimensions are first probed and then noncommutative effects appear continuously beyond the scale Λ_{NC} . So, as in the previous section, we can separate the $\bar{\phi}$ fields into commutative $\bar{\phi}$ fields and noncommutative $\bar{\phi}_{NC}$ fields with the momentum $\Lambda_{NC} < k < \Lambda_0$. From the Wilsonian point of view, the $\bar{\phi}_{NC}$ fields are integrated out when we calculate the effective action. So we get an ordinary ϕ^3 theory compactified on T^2 . The noncommutativity parameter θ appears in renormalized mass and coupling constant with a series of higher order irrelevant operators. On the other hand, if $\Lambda_c \geq \Lambda_{NC}$, the situation will be different. Here,

the extra dimensions are noncommutative at all scales. In this region, the perturbation calculation of nonplanar diagrams shows that the one loop self energy for $\mathbf{n} \neq \mathbf{0}$ gets the form [32],

$$\Sigma = -\frac{g^2}{(4\pi)^3} \left(\frac{R^2}{\theta^2 \mathbf{n}^2} + \frac{5}{24} m^2 \ln \left(\frac{m^2 \theta^2 \mathbf{n}^2}{R^2} \right) + \dots \right) \quad (81)$$

So the one loop effective action S_{eff} contains quadratic and logarithmic IR singularities. While if R is so large that $\Lambda_c < \Lambda_{NC}$, the effective action is free from the UV/IR mixing problem.

5 Conclusion

We have derived a consistent Wilsonian effective field theory for noncommutative ϕ^4 and ϕ^3 theories. The effects of noncommutativity appears in mass, coupling constant and the $C_{n,d,i}$, the coefficients of higher order operators. We have shown that the noncommutative effective ϕ^3 and ϕ^4 theories are free from the UV/IR mixing problem in the limit $\theta \rightarrow 0$. Similarly, all order calculations can also have a soft behaviour and this result is natural in consequence of our assumption that beyond the scale Λ_{NC} , the noncommutative effects can be probed. Below this scale, space-time becomes commutative continuously. We also studied the positivity constraint on the coefficients of the higher order irrelevant operator. These coefficients satisfy this constraint up to one loop explicitly. The noncommutative extra dimensions were considered in the noncommutative ϕ^3 model. Our results show that if the extra dimensions are large such that $\Lambda_c = \frac{1}{R} < \Lambda_{NC}$, the noncommutative IR divergences or UV/IR mixing could disappear. For a similar idea to what is proposed in this paper but in a different context see [26], and also [34].

6 Acknowledgment

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7 Appendix A

The functions $\mathcal{F}(n, m)$ and $\mathcal{G}(n, m; \theta)$ are defined as follows

$$\begin{aligned}\mathcal{F}(n, m) &= \int \frac{d^d p}{(2\pi)^d} \frac{1}{(p^2)^n} \int \frac{d^d q}{(2\pi)^d} \frac{1}{(q^2)^m (q+p)^2} \\ &= \frac{1}{2} \left(\frac{2\pi^{\frac{d}{2}}}{(2\pi)^d \Gamma(\frac{d}{2})} \right)^2 \sum_{r=0}^{\infty} (-1)^r \frac{\Gamma(m+r+1)}{\Gamma(m)r!(r+\frac{d}{2})} \int_0^1 dx \frac{1}{x^m (1-x)^{r+2}} \\ &\quad \times \int_{\Lambda_{NC}}^{\Lambda_0} dp p^{d-2(n+m+r-1)-1} [(\Lambda_0 + xp)^{r+\frac{d}{2}} - (\Lambda_{NC} + xp)^{r+\frac{d}{2}}] \quad (82)\end{aligned}$$

and

$$\begin{aligned}\mathcal{G}(n, m; \theta) &= \int \frac{d^d p}{(2\pi)^d} \frac{1}{(p^2)^n} \int \frac{d^d q}{(2\pi)^d} \frac{\cos p\theta q}{(q^2)^m (q+p)^2} \\ &= \frac{1}{2} \left(\frac{2\pi^{\frac{d}{2}}}{(2\pi)^d \Gamma(\frac{d}{2})} \right)^2 \frac{1}{\Gamma(m)} \sum_{r=0}^{\infty} (-1)^r \frac{1}{r!(r+\frac{d}{2})} \int_0^1 (1-x)^m \int_0^{\infty} d\tau \tau^{m+r} \\ &\quad \times \int_{\Lambda_{NC}}^{\Lambda_0} dp p^{d-2n-1} \exp\left(-\tau x(1-x)p^2 - \frac{1}{4\tau}\tilde{p}^2\right) \\ &\quad \times \left[(\Lambda_0 + \frac{i}{2\tau}\tilde{p} + xp)^{r+\frac{d}{2}} - (\Lambda_{NC} + \frac{i}{2\tau}\tilde{p} + xp)^{r+\frac{d}{2}}\right] \quad (83)\end{aligned}$$

References

- [1] N. Seiberg and E. Witten, *String theory and noncommutative geometry*, JHEP **9909**, 032 (1999), [hep-th/9908142].
- [2] M. R. Douglas and N. A. Nekrasov, *Noncommutative field theory*, Rev. Mod. Phys. **73** (2001) 977, [hep-th/0106048].
- [3] R. J. Szabo, *Quantum field theory on noncommutative space*, Phys. Rept. **378** (2003) 207, [hep-th/0109162].
- [4] A. Micu and M. M. Sheikh-Jabbari, *Noncommutative Φ^4 theory at two loop*, JHEP **0101** (2001) 025 [hep-th/0008057].
- [5] I. Y. Arefeva, D. M. Belov and A. S. Koshelev, *Two-loop diagrams in noncommutative ϕ_4^4 theory*, Phys. Lett. **B476** (2000) 431, [hep-th/9912075].
- [6] A. Armoni, *Comments on perturbation dynamics of noncommutative Yang-mills theory*, Nucl. Phys. **B593** (2001) 229, [hep-th/0005208].

- [7] C. P. Martin and D. Sanchez, *The one-loop UV divergent structure of $U(1)$ Yang-Mills theory on noncommutative R^4* , Phys. Rev. Lett. **83** (1999) 476, [hep-th/9903077].
- [8] L. Griguolo and M. Pietroni, *Wilsonian renormalization group and the noncommutative IR/UV connection*, JHEP **0105** (2001) 032, [hep-th/0104217].
- [9] S. Sarkar, *On the renormalizability of noncommutative field theories*, JHEP **0206** (2002) 003 [hep-th/0202171].
- [10] L. Griguolo and M. Pietroni, *Hard noncommutative loops resummation*, Phys. Rev. Lett. **88** (2002) 071601, [hep-th/0102070].
- [11] S. Minwalla, M. V. Raamsdonk and N. Seiberg, *Noncommutative perturbative dynamics*, JHEP **02** (2000) 020, [hep-th/9912072].
- [12] L. Alvarez-Gaume and M. A. Vazquez-mozo, *General Properties of Noncommutative Field Theories*, Nucl. Phys. **B668**, (2003) 293, [hep-ph/0305093].
- [13] S. A. Abel, J. Jaeckel, V. V. Khoze and A. Ringwald, *Vacuum birefringence as a probe of Planck scale noncommutativity*, [hep-ph/0607188].
- [14] S. Abel, C-S. Chu, M. Godsell, *Noncommutativity from the string perspective: modification of gravity at a mm without mm sized extra dimensions*, [hep-th/0606248].
- [15] B. Mirza, M. Zarei, *Cancellation of soft and collinear divergences in noncommutative QED*, Phys.Rev. **D74** (2006) 065019, [hep-th/0609181].
- [16] H. Grosse and R. Wulkenhaar, *Renormalisation of ϕ^4 -theory on noncommutative R^4 in the matrix base*, Commun. Math. Phys. **256** (2005) 305, [hep-th/0401128].
- [17] H. Grosse and R. Wulkenhaar, *Renormalisation of ϕ^4 -theory on noncommutative R^4 to all orders*, [hep-th/0403232].
- [18] H. Grosse and R. Wulkenhaar, *Noncommutative QFT and Renormalization*, [hep-th/0607208].
- [19] V. Rivasseau and F. Vignes-Tourneret, *Renormalization of Noncommutative field theories*, [hep-th/0702068].

- [20] J. M. CORMONA and J. L. CARTES, *Infrared and ultraviolet cut-offs of quantum field theory*, Phys. Rev. D**65**, (2002) 025006, [hep-ph/0012028].
- [21] A. ADAMS, N. ARKANI-HAMED, S. DUBOUSKY, A. NICOLIS and R. RATTAZI, *Causality, analyticity and an IR obstruction to UV completion*, [hep-th/0602178].
- [22] J. POLCHINSKI, Nucl. phys. B**231** (1984) 269.
- [23] M. BONINI, M. D'ATTANASIO and G. MARCHESINI, *Perturbative renormalization and infrared finiteness in the Wilson renormalization group: the massless scalar case*, Nucl. phys. B**409**, (1993) 441, [hep-th/9301114].
- [24] E. LANGMANN, R. J. SZABO, *Duality in Scalar Field Theory on Noncommutative Phase Spaces*, Phys. Lett. B**533** (2002) 168, [hep-th/0202039].
- [25] R. GURAU, J. MAGNEN, V. RIVASSEAU and F. VIGNES-TOURNERET, *Renormalization of Non-Commutative Φ^4 Field Theory in x Space*, Commun. Math. Phys. **267** (2006) 515, [hep-th/0512271].
- [26] D. BAHNS and S. WALDMANN, *Locally Noncommutative Space-Times*, [math.QA/0607745].
- [27] I. S. GRADSHTEYN and I. M. RYZHIK, Fifth edition, Academic Press, Inc., London, (1994).
- [28] S. W. HAWKING and T. HERTOG, *Living with ghosts*, Phys. Rev. D**65** (2002) 103515, [hep-th/0107088].
- [29] J. Z. SMON, *Higher derivative Lagrangians, nonlocality, problems and solutions*, Phys. Rev. D**41** (1990) 3720.
- [30] I. ANTONIADIS, E. DUDAS, D. M. GHILENCEA, *Living with ghosts and their radiative corrections*, [hep-th/0608094].
- [31] M. V. RAAMSDONK and N. SEIBERG, *Comments on noncommutative perturbation dynamics*, JHEP **0003** (2000) 035, [hep-th/0002186].
- [32] J. GOMIS, T. MEHEN and M. B. WISE, *Quantum field theories with compact non-commutative extra dimensions*, JHEP **0008** (2000) 029, [hep-th/0006160].

- [33] S. A. Abel, J. Jaeckel, V. V. Khoze and A. Ringwald, *Noncommutativity, extra dimensions and power law running in the infrared*, JHEP **0601** (2006) 105, [hep-ph/0511197].
- [34] Chong-Sun Chu, Brian R. Greene, Gary Shiu, *Remarks on Inflation and Noncommutative Geometry*, Mod. Phys. Lett **A 16** (2001) 2231-2240, [hep-th/0011241].